Simple Harmonic Motion

Periodic motion is motion of an object that represents the object returns to a given position after a fixed time interval. Examples of oscillatory motion are :

oscillating pendulum, vibrations of a stretched string, vibration of electrons, and movement of light in a laser beam

- Consider a particle located at point P on the circumference of a circle of radius A, with the line OP making an angle δ with the x axis at $t = 0$.
- If the particle moves along the circle with constant angular velocity ω the angle between OP and the x axis is $(\theta + \delta)$.
- As the particle moves along the circle, the projection of OP on the y axis, labeled point Q as in Fig. moves back and forth along the y axis between the limits $y = \pm A$.
- From the triangle OPQ, we see that
- $y = A \sin(\omega t+\delta)$

Where $\theta = \omega t$ and y is called the displacement of the vibrating particle. The rate of change of displacement is called the velocity of the vibrating particle.

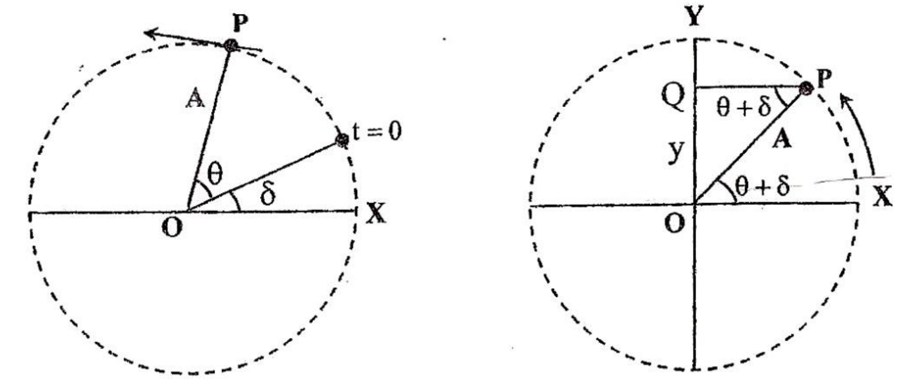

Figure (1a) Figure (1b)

The velocity is given as,

$$\dot{y} = \frac{dy}{dt} = A\omega\cos(\omega t + \delta)$$

The acceleration is given by.

$$\ddot{y} = \frac{d^2y}{dt^2} = -A\omega^2\sin(\omega t + \delta) = -\omega^2 y$$

The last equation shows that the acceleration $\ddot{y}$ is directly proportional to the displacement y and in opposite direction to it.
In this case, the object moves with **simple harmonic motion (SHM)**.

- We can summarize this type of motion in the following.

Displacement	$y = A \sin(\omega t + \delta)$
Velocity	$\dot{y} = \frac{dy}{dt} = A\omega \cos(\omega t + \delta)$
Acceleration	$\ddot{y} = \frac{d^2y}{dt^2} = -A\omega^2 \sin(\omega t + \delta) = -\omega^2 y$
Force	$F = -m\omega^2 y$
Amplitude	A
Maximum velocity	$A\omega$
Maximum acceleration	$A\omega^2$
Period	$T = 2\pi/\omega$

Energy of the simple harmonic oscillator .

The kinetic energy of the particle is given by:

$$K.E = \frac{1}{2}\, m\upsilon^2 = \frac{1}{2}\, m\dot{y}^2$$

The potential energy of the vibrating particle is the amount of work done in moving the particle a distance y. So,

$$P.E = \int_0^y -F(y)dy$$

$$= \int_0^y -(-m\omega^2\, y)dy$$

$$= m\omega^2 \int_0^y y\, dy$$

$$= \frac{1}{2} m\omega^2 y^2$$

$$E = \frac{1}{2}\, m\dot{y}^2 + \frac{1}{2} m\omega^2 y^2$$

$\square$ $\square$

$$E = \frac{1}{2}\, m\omega^2 A^2 \cos^2 \omega(t + \square) + \frac{1}{2}\, m\omega^2 A^2 \sin^2 \omega(t + \quad)$$

$$= \frac{1}{2}\, m\omega^2 A^2 [\cos^2 \omega(t + \quad) + \sin^2 \omega(t + \quad)]$$

$$= \frac{1}{2}\, m\omega^2 A^2$$

The Simple Pendulum

- F = -mg sin θ, sin θ = $\frac{x}{\ell}$
- F = -mg $\frac{x}{\ell}$
- From Newton's second law
- F = m$\ddot{x}$
- m$\ddot{x}$ = -mg $\frac{x}{\ell}$
- $\ddot{x}$ = - $\frac{g}{\ell}$x
- This equation is similar to the equation of simple harmonic motion
- $\ddot{y} = -\omega^2 y$
- Comparing the last two equations:
- $\omega^2 = \frac{g}{\ell}$
- So the angular velocity is given by:
- $\omega = \sqrt{\frac{g}{l}}$ and the period of the motion is:
- $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$

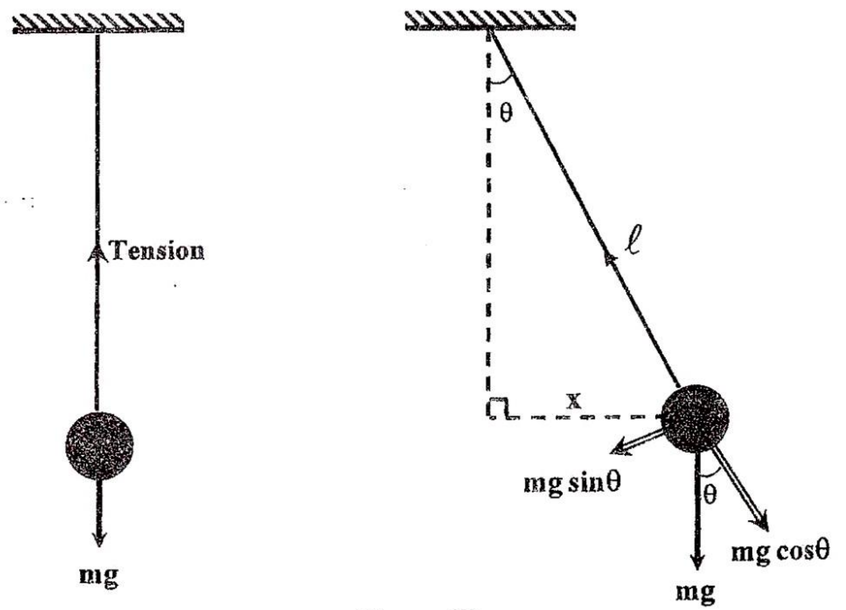

Figure (3)

Oscillating spring

- From Hook's law:

  F = - k x

- $m\ddot{x} = - k x$

- $\ddot{x} = -\frac{k}{m}x$

- This equation is like the equation of simple harmonic motion

- $\ddot{y} = -\omega^2 y$

- Comparing the last two equations:

- $\omega = \sqrt{\frac{k}{m}} \quad \rightarrow \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$

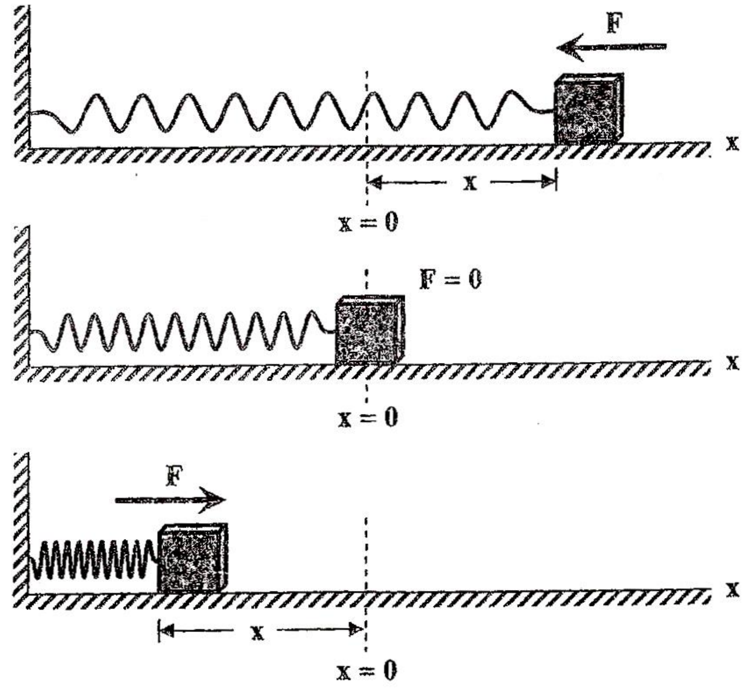

Figure (4)